

Itô Formula for Processes Taking Values in Intersection of Finitely Many Banach Spaces

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Abstract Motivated by applications to SPDEs we extend the Itô formula for the square of the norm of a semimartingale $y(t)$ from [6] to the case

$$\sum_{i=1}^m \int_{(0,t]} v_i^*(s) dA(s) + h(t) =: y(t) \in V \quad dA \times \mathbb{P}\text{-a.e.},$$

where A is an increasing right-continuous adapted process, v_i^* is a progressively measurable process with values in V_i^* , the dual of a Banach space V_i , h is a cadlag martingale with values in a Hilbert space H , identified with its dual H^* , and $V := V_1 \cap V_2 \cap \dots \cap V_m$ is continuously and densely embedded in H .

The formula is proved under the condition that $\|y\|_{V_i}^{p_i}$ and $\|v_i^*\|_{V_i^*}^{q_i}$ are almost surely locally integrable with respect to dA for some conjugate exponents p_i, q_i . This condition is essentially weaker than the one which would arise in application of the results in [6] to the semimartingale above.

Keywords Stochastic evolution equations, Stochastic partial differential equations, Itô's formula, Energy equality

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1 Introduction

Itô formula for the square of the norm is an essential tool in the study of stochastic evolution equations of the type

$$dv(t) = \mathbb{A}(t, v(t)) dt + \sum_k \mathbb{B}_k(t, v(t)) dW^k(t), \quad (1.1)$$

where $(W^k)_{k=1}^\infty$ is a sequence of independent Wiener processes, and $\mathbb{A}(t, \cdot)$ and $\mathbb{B}_k(t, \cdot)$ are (possibly random nonlinear) operators on a separable real Banach space V , with values in a Banach space V' and a Hilbert space H respectively, such that $V \hookrightarrow H \hookrightarrow V'$ with continuous and dense embeddings. We assume there is a constant K such that $(v, h) \leq K\|v\|_V\|h\|_{V'}$ for all $v \in V$ and $h \in H$. This means that for the linear mapping $\Psi : H \rightarrow H^*$, which identifies H with its dual H^* via the inner product in H , we have $\|\Psi(h)\|_{V^*} \leq K\|h\|_{V'}$. Therefore, since H is dense in V' , Ψ can be extended to a continuous mapping from V' into V^* , the dual of V . It is assumed that this extension is one-to-one from V' into V^* . Thus an initial value problem for equation (1.1) can be viewed as

$$v(t) = \int_0^t v^*(s) ds + h(t) =: y(t) \quad (1.2)$$

with the V^* -valued process $v^*(t) := \mathbb{A}(t, v(t))$ and $H \equiv H^*$ -valued process

$$h(t) := h_0 + \sum_k \int_0^t \mathbb{B}_k(s, v(s)) dW^k(s),$$

where h_0 is a given initial value and the equality (1.2) in V^* is required $dt \times \mathbb{P}$ almost everywhere. In the special case $B_k = 0$ for every k , and nonrandom h_0 and A , i.e., in the case

$$v(t) = h_0 + \int_0^t v^*(s) ds, \quad dt\text{-a.e.},$$

it is well-known that when $v \in L_p([0, T], V)$, $v^* \in L_q([0, T], V^*)$ for $T > 0$ and conjugate exponents p and q , then there is $u \in C([0, T], H)$ such that $u = v$ for dt -almost all $t \in [0, T]$ and the “energy equality”

$$|u(t)|_H^2 = |h_0|_H^2 + 2 \int_0^t \langle v^*(s), v(s) \rangle ds$$

holds for all $t \in [0, T]$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of V^* and V . This formula is used in proofs of existence and uniqueness theorems for PDEs, see e.g., [2] and [13]. A generalisation of it, a “stochastic energy equality”, i.e., an Itô formula for the square of the H -norm of y , was first presented in Pardoux [14], and was used to obtain existence and uniqueness theorems for SPDEs. The proof of it in [14] was not separated from the theory of SPDEs developed there. A proof, not bound to the theory of SPDEs, was given in Krylov and

Rozovskii [12], and then this stochastic energy equality was generalised in Gyöngy and Krylov [6] to V^* -valued semimartingales y of the form

$$y(t) = \int_{(0,t]} v^*(s) dA(s) + h(t), \quad (1.3)$$

where A is an adapted nondecreasing cadlag process and h is an H -valued cadlag martingale. This generalisation is used in Gyöngy [7] to extend the theory of SPDEs developed in [14] and [12] to SPDEs driven by random orthogonal measures and Lévy martingales, written in the form

$$dv(t) = \mathbb{A}(t, v(t)) dA(t) + \mathbb{B}(t, v(t)) dM(t) \quad (1.4)$$

with cadlag (quasi left-continuous) martingales M with values in a Hilbert space.

In the present paper we are interested in stochastic energy equalities which can be applied to SPDEs (1.4) when \mathbb{A} is of the form $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2 + \dots + \mathbb{A}_m$ and the operators \mathbb{A}_i have different analytic and growth properties. This means,

$$\mathbb{A}_i(t, \cdot) : V_i \rightarrow V_i' \quad i = 1, 2, \dots, m$$

for some Banach spaces V_i and V_i' , such that with a constant R and a process g , locally integrable with respect to dA , one has for all t

$$\|\mathbb{A}_i(t, w)\|_{V_i'} \leq |g_t|^{1/q_i} + R\|w\|_{V_i}^{p_i-1}$$

for all $w \in V$, $q_i = p_i/(p_i - 1)$ with (possibly) different exponents $p_i \geq 1$, which for $p_i = 1$ means that $\|\mathbb{A}_i(t, w)\|_{V_i'}$ is bounded by a constant.

In the special case when $A(t) = t$ and M is a Wiener process the above situation was considered in [14], and a related stochastic energy equality was also presented there. Our main result, Theorem 2.1 generalises the results on stochastic energy equalities from [14] and [6]. We prove it by adapting the method of the proof of the main theorem in [6].

In the present paper we consider a semimartingale y of the form (1.3) such that $dA \times \mathbb{P}$ -almost everywhere y takes values in $V = V_1 \cap \dots \cap V_m$, where V_i are Banach spaces (over \mathbb{R}) such that V with the norm $\|\cdot\| := \sum_{i=1}^m \|\cdot\|_{V_i}$ is continuously and densely embedded in H . The process v^* in (1.3) is of the form $v^* = \sum_{i=1}^m v_i^*$, where v_i^* are V_i^* -valued progressively measurable processes. We prove that y is almost surely cadlag as a process with values in H and for $|y|_H^2$ an Itô formula holds under the assumption that $\|y\|_{V_i}^{p_i}$ and $\|v_i^*\|_{V_i^*}^{q_i}$ are almost surely locally integrable with respect to dA for some conjugate exponents p_i, q_i . See Section 2 for precise formulation of the main theorem. To apply the result of [6] to y given by (1.3), one needs the local integrability (with respect to dA) of

$$\|y\|_V \|v^*\|_{V^*} = (\|y\|_{V_1} + \dots + \|y\|_{V_m}) \|v_1^* + \dots + v_m^*\|_{V^*},$$

which, in general, is not satisfied under our assumptions. See Remark 2.1 and Example 2.1.

We note that in the context of stochastic evolution equations it is possible to prove Itô formulae for more general functions (satisfying appropriate differentiability assumptions), see again Pardoux [14], Krylov [11], [10], [9], Da Prato, Jentzen and Röckner [4], as well as Dareiotis and Gyöngy [1]. The Itô formula for the square of the norm is used in particular to establish a priori estimates as well as uniqueness and existence of solutions of stochastic evolution equations. The more general Itô formula can then be used to study finer properties of solutions of stochastic evolution equations, for example the maximum principle.

For general theory of SPDEs in the variational setting we refer the reader to Krylov and Rozovskii [12], Prévôt and Röckner [15] and Rozovskii [16].

2 Main Results

For $i = 1, \dots, m$ let $(V_i, \|\cdot\|_{V_i})$ be real Banach spaces with duals $(V_i^*, \|\cdot\|_{V_i^*})$. Let V denote the vector space $V_1 \cap \dots \cap V_m$ with the norm $\|\cdot\| := \|\cdot\|_{V_1} + \dots + \|\cdot\|_{V_m}$. Then clearly, V is a Banach space. Assume that it is separable and is continuously and densely embedded in a Hilbert space $(H, |\cdot|)$, which is identified with its dual H^* by the help of the inner product (\cdot, \cdot) in H . Thus we have

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where $H^* \hookrightarrow V^*$ is the adjoint of the embedding $V \hookrightarrow H$. We use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V^* . Note that if $v^* \in V_i^*$ for some i , then its restriction to V belongs to V^* and $|\langle v^*, v \rangle| \leq \|v^*\|_{V_i^*} \|v\|_{V_i}$ for all $v \in V$. Note also that $\langle v^*, v \rangle = (h, v)$ for all $v \in V$ when $v^* = h \in H$.

A complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with an increasing family of σ -algebras $(\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_t \subset \mathcal{F}$ will be used throughout the paper. Moreover it is assumed that the usual conditions are satisfied: $\bigcap_{s > t} \mathcal{F}_s = \mathcal{F}_t$ and \mathcal{F}_0 contains all subsets of \mathbb{P} -null sets of \mathcal{F} . We use the notation $\mathcal{B}(\mathbb{R}_+)$ for the σ -algebra of Borel subsets of $\mathbb{R}_+ = [0, \infty)$, and for a real-valued increasing $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ -measurable process $(A(t))_{t \geq 0}$ the notation $dA \times \mathbb{P}$ stands for the measure defined on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ by

$$(dA \times \mathbb{P})(F) = \mathbb{E} \int_0^\infty \mathbf{1}_F dA(t), \quad F \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Let $h = (h(t))_{t \geq 0}$ be an H -valued locally square integrable martingale that is cadlag (continuous from the right with left-hand limits) in the strong topology on H . Its quadratic variation process is denoted by $[h]$, and $\langle h \rangle$ denotes the unique predictable process starting from zero such that $|h|^2 - \langle h \rangle$ is a local martingale. Furthermore let A be a real-valued nondecreasing adapted cadlag process starting from zero. Finally let $v = (v(t))_{t \geq 0}$ be a V -valued progressively measurable process and for $i = 1, \dots, m$ let $v_i^* = (v_i^*(t))_{t \geq 0}$ be V_i^* -valued processes such that $\langle \varphi, v_i^* \rangle$ are progressively measurable for any $\varphi \in V$. Notice that v is also progressively measurable as a process with values in \bar{V}_i , the closure in V_i -norm of the linear hull of $\{v(t) : t \geq 0, \omega \in \Omega\}$.

Let there be $p_i \in [1, \infty)$ and $q_i = p_i/(p_i - 1) \in (1, \infty]$, where, as usual, $1/0 := \infty$. Assume that for each $i = 1, 2, \dots, m$ and $T > 0$

$$\int_0^T \|v(t)\|_{V_i}^{p_i} dA(t) < \infty, \quad \left(\int_0^T \eta_i^{q_i}(t) dA(t) \right)^{1/q_i} < \infty, \quad (2.1)$$

for some progressively measurable process η_i such that $\|v_i^*\|_{V_i^*} \leq \eta_i$ for $dA \times \mathbb{P}$ -almost everywhere, where for $q_i = \infty$ the second expression means

$$dA\text{-ess sup}_{t \leq T} \eta_i(t),$$

the essential supremum (with respect to dA) of η_i over $[0, T]$.

The following theorem is the main result of this paper.

Theorem 2.1 *Let τ be a stopping time. Suppose that for all $\varphi \in V$ and for $dA \times \mathbb{P}$ almost all (ω, t) such that $t \in (0, \tau(\omega))$ we have*

$$(v(t), \varphi) = \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), \varphi \rangle dA(s) + (h(t), \varphi). \quad (2.2)$$

Then there is $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ and an H -valued cadlag process \tilde{v} such that the following statements hold.

- (i) *For $dA \times \mathbb{P}$ almost all (t, ω) satisfying $t \in (0, \tau(\omega))$ we have $\tilde{v} = v$.*
- (ii) *For all $\omega \in \tilde{\Omega}$ and $t \in [0, \tau(\omega))$ we have*

$$(\tilde{v}(t), \varphi) = \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), \varphi \rangle dA(s) + h(t)\varphi \quad \text{for all } \varphi \in V. \quad (2.3)$$

- (iii) *For all $\omega \in \tilde{\Omega}$ and $t \in [0, \tau(\omega))$*

$$\begin{aligned} |\tilde{v}(t)|^2 &= |h(0)|^2 + 2 \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), v(s) \rangle dA(s) + 2 \int_{(0,t]} (\tilde{v}(s-) dh(s)) \\ &\quad - \int_{(0,t]} |v^*(s)|^2 \Delta A(s) dA(s) + [h]_t, \end{aligned} \quad (2.4)$$

where $v^*(t) := \sum_{i=1}^m v_i^*(t) \in H$ for $\Delta A(t) > 0$.

Consider now a situation where the assumptions on h and A are as above but $m = 1$ and regarding v and $v^* := v_1^*$ we know that $\|v(t)\|$, $\|v^*(t)\|_{V^*}$ and $\|v(t)\| \|v^*(t)\|_{V^*}$ are almost surely locally integrable with respect to $dA(t)$. Let

$$\bar{v}^*(t) := \frac{v^*(t)}{1 + \|v^*(t)\|_{V^*}} \quad \text{and} \quad \bar{A}(t) := \int_{(0,t]} (1 + \|v^*(t)\|_{V^*}) dA(t).$$

Then $\|\bar{v}^*\|_{V^*} \leq 1$ and so v , \bar{v}^* and \bar{A} satisfy the conditions on v , v^* and A , respectively, with $p_1 = 1$ and $q_1 = \infty$. If (2.2) holds for all $\varphi \in V$ and for $dA \times \mathbb{P}$ almost all (ω, t) such that $t \in (0, \tau(\omega))$ then

$$(v(t), \varphi) = \sum_{i=1}^m \int_{(0,t]} \langle \bar{v}^*(s), \varphi \rangle d\bar{A}(s) + (h(t), \varphi).$$

Applying Theorem 2.1 then means that we have all of its conclusions with \bar{v}^* and \bar{A} in place of v^* and A respectively. In particular, we get

$$\begin{aligned} |\tilde{v}(t)|^2 &= |h(0)|^2 + 2 \int_{(0,t]} \langle \bar{v}^*(s), v(s) \rangle d\bar{A}(s) + 2 \int_{(0,t]} (\tilde{v}(s-), dh(s)) \\ &\quad - \int_{(0,t]} |\bar{v}^*(s)|^2 \Delta \bar{A}(s) d\bar{A}(s) + [h]_t \\ &= |h(0)|^2 + 2 \int_{(0,t]} \langle v^*(s), v(s) \rangle dA(s) + 2 \int_{(0,t]} (\tilde{v}(s-), dh(s)) \\ &\quad - \int_{(0,t]} |v^*(s)|^2 \Delta A(s) dA(s) + [h]_t. \end{aligned}$$

Hence we see that Theorem 2.1 is a generalisation of the main theorem in Gyöngy and Krylov [6].

Remark 2.1 One might think that Theorem 2.1 follows from the main theorem in [6] by considering the process $v^* = \sum_i v_i^*$ as a process with values in V^* . However, taking into account that for any $w^* \in V^*$

$$\|w^*\|_{V^*} = \inf \left\{ \max_{i=1,\dots,m} \|w_i^*\|_{V_i^*} : w^* = \sum_{i=1}^m w_i^*, w_i^* \in V_i^* \right\}$$

(see for example Gajewski, Gröger and Zacharias [3, Chapter 1, Theorem 5.13]), one can show that the local integrability condition in [6] for

$$\|v\|_V \|v^*\|_{V^*} = (\|v\|_1 + \dots + \|v\|_m) \|v^*\|_{V^*}$$

is not implied by our assumption (2.1). Thus the main theorem in [6] is not applicable in our situation.

We consider the following motivating example.

Example 2.1 Consider the stochastic partial differential equation

$$\begin{aligned} du &= [\nabla(|\nabla u|^{p_1-2} \nabla u) + |u|^{p_2-2} u] dt \\ &\quad + f(u, \nabla u) dW + \int_Z g(u) q(dt, dz) \text{ in } \mathcal{D} \times (0, T). \end{aligned}$$

Here W is a Wiener process (finite or infinite dimensional depending on the choice of f), (Z, Σ) is a measurable space and $q(ds, dz)$ a stochastic martingale

measure on $[0, \infty) \times Z$. See, for example, Gyöngy and Krylov [5] for detailed definition. We take \mathcal{D} to be a bounded Lipschitz domain in \mathbb{R}^d .

It is natural to assume that a solution u should be such that $\|u\|_{W_{p_1}^{p_1}(\mathcal{D})}$ and $\|u\|_{L_{p_2}^{p_2}(\mathcal{D})}$ are almost surely locally integrable. To apply the result in Gyöngy and Krylov [6] one could try to take $V := W_{p_1}^1(\mathcal{D}) \cap L_{p_2}(\mathcal{D})$ with the norm $\|\cdot\|_V = \|\cdot\|_{W_{p_1}^1(\mathcal{D})} + \|\cdot\|_{L_{p_2}(\mathcal{D})}$. The dual of V can be identified with the linear space

$$V^* = \{f = f_1 + f_2 : f_1 \in W_{p_1}^1(\mathcal{D})^*, f_2 \in L_{p_2}(\mathcal{D})^*\}$$

equipped with the norm

$$\begin{aligned} \|f\|_{V^*} &= \inf\{\max(\|f_1\|_{W_{p_1}^1(\mathcal{D})^*}, \|f_2\|_{L_{p_2}(\mathcal{D})^*}) : \\ &\quad f = f_1 + f_2, f_1 \in W_{p_1}^1(\mathcal{D})^*, f_2 \in L_{p_2}(\mathcal{D})^*\}. \end{aligned}$$

One would then need to show that $\|u\|_V \|\nabla(|\nabla u|^{p_1-2} \nabla u) + |u|^{p_2-2} u\|_{V^*}$ is locally integrable. To ensure this in general we need, in particular, that

$$\|u\|_{W_{p_1}^1(\mathcal{D})} \| |u|^{p_2-2} u \|_{L_{p_2}(\mathcal{D})^*} = \|u\|_{W_{p_1}^1(\mathcal{D})} \|u\|_{L_{p_2}^{p_2-1}(\mathcal{D})}$$

is locally integrable, which we may not have if $p_1 < p_2$. Thus one cannot apply the Itô formula from Gyöngy and Krylov. On the other hand it is easy to check that the assumptions of Theorem 2.1 are satisfied.

An application of the above Itô's formula to SPDEs driven by Wiener processes is given in [14] (Chapter 2, Example 5.1) and in [8]. Further examples can be found in [13, Chapter 2, Section 1.7].

3 Preliminaries

Lemma 3.1 *For $r \in [0, \infty)$ let $\beta(r) := \inf\{t \geq 0 : A(t) \geq r\}$ and let $x(t)$ be a real valued process that is locally integrable with respect to dA for all $\omega \in \Omega$. Then*

- i) $\beta(r)$ is a stopping time (not necessarily finite) for every $r \in [0, \infty)$,
- ii)

$$\begin{aligned} \int_{(0,t]} x(s) dA(s) &= \int_{(0,A(t)]} x(\beta(r)) dr, \\ \int_{(0,t)} x(s) dA(s) &= \int_{(0,A(t-)]} x(\beta(r)) dr \end{aligned}$$

- for every $t \in [0, \infty)$,
- iii)

$$A(\beta(t)-) - A(\beta(s)) \leq t - s$$

for every $s, t \in [0, \infty)$.

iv) If $0 = r_0^n < r_1^n < \dots < r_k^n < \dots$ is an increasing sequence of decompositions of $[0, \infty)$ such that $\sup_k |r_{k+1}^n - r_k^n| \rightarrow 0$ as $n \rightarrow \infty$ then for every $t \geq 0$ and $\omega \in \Omega$

$$\sum_k |X(\tau_{k+1}^n \wedge t) - X(\tau_k^n \wedge t)|^2 \rightarrow \sum_{s \leq t} |X(s)|^2 |\Delta A(s)|^2$$

as $n \rightarrow \infty$, where $X(t) := \int_{(0,t]} x(s) dA(s)$ and $\tau_k^n := \beta(r_k^n)$.

This Lemma is proved in Gyöngy and Krylov [6, Lemma 1].

Let $\kappa_n^{(j)}$ for $j = 1, 2$ and integers $n \geq 1$ denote the functions defined by

$$\kappa_n^{(1)}(t) = 2^{-n} \lfloor 2^n t \rfloor, \quad \kappa_n^{(2)}(t) = 2^{-n} \lceil 2^n t \rceil$$

The following lemma is known and the authors believe is due to Doob.

Lemma 3.2 For integers $i \geq 1$ let $(X_i, \|\cdot\|_{X_i})$ be Banach spaces, and let $p_i \in [1, \infty)$. Let $x_i : \mathbb{R} \times \Omega \rightarrow X_i$ be $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ Bochner-measurable such that $x_i(r) = 0$ for $r \notin [0, 1]$ and

$$\alpha_i := \mathbb{E} \int_0^1 \|x_i(r)\|_{X_i}^{p_i} dr < \infty.$$

Then there exists a subsequence $n_k \rightarrow \infty$ such that for dt -almost all $t \in [0, 1]$

$$\mathbb{E} \int_{(0,1]} \|x_i(r) - x_i(\kappa_{n_k}^{(j)}(r-t) + t)\|_{X_i}^{p_i} dr \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

for $j = 1, 2$ and all $i \geq 1$.

Proof. Let $(c_i)_{i=1}^\infty$ be a sequence of positive numbers such that

$$\sum_{i=1}^\infty c_i 2^{p_i} \alpha_i < \infty.$$

By change of variables and changing the order of integration

$$\begin{aligned} I_n &:= \sum_{i=1}^\infty c_i \int_0^1 \mathbb{E} \int_0^1 \|x_i(r) - x_i(\kappa_n^{(j)}(r-t) + t)\|_{X_i}^{p_i} dr dt \\ &\leq \sum_{i=1}^\infty c_i \mathbb{E} \int_{-1}^1 \int_0^1 \|x_i(s+t) - x_i(\kappa_n^{(j)}(s) + t)\|_{X_i}^{p_i} dt ds. \end{aligned}$$

Note that by the shift invariance of the Lebesgue measure

$$J_{in}(s) := \int_0^1 \|x_i(s+t) - x_i(\kappa_n^{(j)}(s) + t)\|_{X_i}^{p_i} dt \rightarrow 0 \quad (a.s.)$$

for $s \in (0, 1)$, $i \geq 1$, and

$$\sum_{i=1}^\infty c_i |J_{in}(s)| \leq \sum_{i=1}^\infty c_i 2^{p_i-1} \left(\int_0^1 \|x_i(s+t)\|_{X_i}^{p_i} dt + \int_0^1 \|x_i(\kappa_n(s) + t)\|_{X_i}^{p_i} dt \right)$$

$$\leq \sum_{i=1}^{\infty} c_i 2^{p_i} \int_0^1 \|x_i(t)\|_{X_i}^{p_i} dt.$$

Therefore by Lebesgue's theorem on dominated convergence

$$I_n = \int_0^1 \left(\sum_{i=1}^{\infty} c_i \mathbb{E} \int_0^1 \|x_i(r) - x_i(\kappa_n^{(j)}(r-t) + t)\|_{X_i}^{p_i} dr \right) dt \rightarrow 0.$$

Hence for a subsequence $n_k \rightarrow \infty$

$$\sum_{i=1}^{\infty} c_i \mathbb{E} \int_0^1 \|x_i(r) - x_i(\kappa_n^{(j)}(r-t) + t)\|_{X_i}^{p_i} dr \rightarrow 0$$

for almost all $t \in [0, 1]$, and the statement of the lemma follows. \square

The following lemma is proved in Gyöngy and Krylov [6, Lemma 3].

Lemma 3.3 *Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of H -valued predictable processes. Suppose*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}, t \leq T} |\xi_n(t)| < \infty \right] = 1$$

and

$$\mathbb{P} \left[\forall t \leq T, \forall \varphi \in H \quad \lim_{n \rightarrow \infty} (\xi_n(t), \varphi) = 0 \right] = 1.$$

Then for any $\varepsilon > 0$

$$\mathbb{P} \left[\sup_{t \leq T} \left| \int_{(0,t]} (\xi_n(s), dh(s)) \right| > \varepsilon \right] \rightarrow 0$$

as $n \rightarrow \infty$.

4 Proof of the Main Result

The following standard steps, as in Krylov and Rozovskii [12], allow us to work under more convenient assumptions without any loss of generality.

- 1) We note that τ can be assumed to be a bounded stopping time. Indeed if we prove Theorem 2.1 under this assumption then we can extend it to unbounded stopping times by considering $\tau \wedge n$ and letting $n \rightarrow \infty$. In fact using a non-random time change we may assume that $\tau \leq 1$.
- 2) Recall the processes η_i from assumption (2.1), and set

$$Q_i(t) = \left(\int_{(0,t]} \eta_i^{q_i}(s) dA(s) \right)^{1/q_i} \quad t \geq 0$$

when $q_i < \infty$, and for $q_i = \infty$ let $Q_i = (Q_i(t))_{t \geq 0}$ denote a nondecreasing cadlag adapted process such that almost surely

$$dA\text{-ess sup}_{s \leq t} \eta_i(s) \leq Q_i(t) \quad \text{for all } t \geq 0.$$

It is not difficult to see that such a process Q_i exists, we can take, e.g., the adapted right-continuous modification of the process $dA\text{-ess sup}_{s \leq t} \eta_i(s)$, i.e.,

$$\lim_{n \rightarrow \infty} dA\text{-ess sup}_{s \leq t+1/n} \eta_i(s).$$

Let $(e^j)_{j \in \mathbb{N}} \subset V$ be an orthonormal basis in H and define

$$\begin{aligned} r(t) := & |h(0)| + A(t) + \sum_{i=1}^m \left(\int_{(0,t]} \|v(s)\|_{V_i}^{p_i} dA(s) \right)^{1/p_i} \\ & + \sum_{i=1}^m Q_i(t) + \sum_{i=1}^m \sum_{k \in \mathbb{N}} 2^{-c_k} \left(\int_{(0,t]} \|w_k(s)\|_{V_i}^{p_i} dA(s) \right)^{1/p_i}, \end{aligned} \quad (4.1)$$

with $c_k := \max_{1 \leq i \leq m} \sum_{j \leq k} |e_j|_{V_i}^2$ and $w_k := \Pi^k h$, where Π^k denotes the orthogonal projection of H onto its subspace spanned by $(e_i)_{i=1}^k$. We may and will assume, without loss of generality, that r and $\langle h \rangle$ are bounded. Indeed, imagine we have proved Theorem 2.1 under this assumption. Consider

$$\tau_n := \inf\{t \geq 0 : r(t) \geq n\}.$$

Then τ_n is a stopping time and $\tau_n \rightarrow \infty$ for $n \rightarrow \infty$. Since $\langle h \rangle$ is a predictable process starting from 0, there is an increasing sequence of stopping times σ_n such that $\sigma_n \rightarrow \infty$ and $\langle h \rangle_t \leq n$ for $t \in [0, \sigma_n]$. Therefore $\tau_n \wedge \sigma_n \wedge \tau \rightarrow \tau$ as $n \rightarrow \infty$, and for fixed n we get $r(t) \leq n$ for $t \in (0, \tau_n \wedge \sigma_n)$ and $\langle h \rangle_t \leq n$ for $t \in [0, \tau_n \wedge \sigma_n]$. Thus we get (2.3) and (2.4) for the stopping time $\tau_n \wedge \sigma_n \wedge \tau$ in place of τ . Letting $n \rightarrow \infty$ provides (2.3) and (2.4) for τ . Thus we may assume that there is $n \geq 1$ such that $r(t) \leq n$ for $t \in (0, \tau)$ and $\langle h \rangle_t \leq n$ for $t \in [0, \tau]$. Moreover, by taking $h\mathbf{1}_{|h(0)| < n}$, $v\mathbf{1}_{|h(0)| < n}$ and $A\mathbf{1}_{|h(0)| < n}$ in place of h , v and A , respectively, and then taking $n \rightarrow \infty$, we may assume that $r(t) \leq n$ for $t \in [0, \tau)$ and $\langle h \rangle_t \leq n$ for $t \in [0, \tau]$. Furthermore, we can define $A(t) := A(\tau-)$, $h(t) = h(\tau)$, $v(t) = 0$ and $v_i^*(t) = 0$ for $t \geq \tau$. Then $r(t) \leq n$ and $\langle h \rangle_t \leq n$ for $t \in [0, \infty)$.

- 3) Finally, we can assume that $r(t) \leq 1$ for $t \in [0, \tau)$ and $\langle h \rangle_t \leq 1$ for $t \in [0, \tau]$. Indeed let $v_n := n^{-1}v$, $A_n := n^{-1}A$ and $h_n := n^{-1}h$. Then r_n , defined analogously to r in (4.1) but with v , A and h replaced by v_n , A_n and h_n respectively, satisfies $r_n(t) \leq n^{-1}r(t) \leq 1$. We thus get (2.3) and (2.4) with v , A and h replaced by v_n , A_n and h_n respectively. We can now multiply by n and n^2 to obtain the desired conclusions.

Now we proceed to prove Theorem 2.1 under the assumption that $\tau \leq 1$, $r(t) \leq 1$ and $\langle h \rangle_t \leq 1$ for $t \in [0, \infty)$. Our approach is the same as in Gyöngy and Krylov [6]. The idea is to approximate v by simple processes whose jumps happen at stopping times where equation (2.2) holds. But (2.2) only holds for every $\varphi \in V$ and $dA \times \mathbb{P}$ almost all $(t, \omega) \in]0, \tau[$, and thus it is not immediately clear how to choose an appropriate piecewise constant approximation to v . Here and later on for stopping times τ the notation $]0, \tau[$ means the stochastic interval $\{(t, \omega) : t \in (0, \tau(\omega)), \omega \in \Omega\}$.

Proposition 4.1 *There is a nested sequence of random partitions of $[0, \infty]$,*

$$0 = \tau_0^n < \tau_1^n \leq \tau_2^n \leq \dots \leq \tau_{N(n)+1}^n = \infty,$$

with stopping times τ_j^n , $j = 1, \dots, N(n) + 1$, such that for every $\omega \in \Omega$ either $\tau_j^n(\omega) < \tau(\omega)$ or $\tau_j^n(\omega) = \infty$, and such that the following statements hold.

(1) *There is $\Omega' \subset \Omega$ such that $\mathbb{P}(\Omega') = 1$ and with*

$$I(\omega) := \{\tau_j^n(\omega) : n \in \mathbb{N}, j = 1, \dots, N(n)\} \cap (0, \infty)$$

we have (2.2) satisfied for every $\omega \in \Omega'$, $t \in I(\omega)$ and $\varphi \in V$. Moreover, if $\Delta A(t) > 0$ for some $t > 0$ and $\omega \in \Omega'$, then $t \in I(\omega)$. Furthermore, if $0 \leq s < t$ and $(s, t] \cap I(\omega) = \emptyset$, then $A(s) = A(t)$.

(2) *For $l \in \{1, 2\}$, $i = 1, \dots, m$ and for all $k \geq 1$*

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \int_{(0, \infty)} \|v(s) - v_n^{(l)}(s)\|_{V_i}^{p_i} dA(s) &= 0, \\ \lim_{n \rightarrow \infty} \mathbb{E} \int_{(0, \infty)} \|w_k(s) - w_{kn}^{(l)}(s)\|_{V_i}^{p_i} dA(s) &= 0, \end{aligned} \quad (4.2)$$

where

$$v_n^{(1)}(t) := \sum_{j=1}^{N(n)} v(\tau_j^n) \mathbf{1}_{[\tau_j^n, \tau_{j+1}^n)}(t), \quad v_n^{(2)}(t) := \sum_{j=0}^{N(n)} v(\tau_{j+1}^n) \mathbf{1}_{(\tau_j^n, \tau_{j+1}^n]}(t),$$

and $w_{kn}^{(l)}$ is defined analogously from $w_k = \Pi^k h$.

Proof. Since V is separable there is $\{\varphi_i\}_{i \in \mathbb{N}} \subset V$ which is dense in V . For each φ_i there is an exceptional set $D_i \in [0, \infty) \times \Omega$ such that (2.2) holds for $(t, \omega) \in \llbracket 0, \tau \rrbracket \setminus D_i$ and $(dA \times \mathbb{P})(D_i) = 0$. Let $D = \bigcup_{i \in \mathbb{N}} D_i$. Then $(dA \times \mathbb{P})(D) = 0$ and (2.2) holds for all $\varphi \in V$ and all $(t, \omega) \in \llbracket 0, \tau \rrbracket \setminus D$. Now using Lemma 3.1 and the Fubini theorem

$$\begin{aligned} 0 &= \mathbb{E} \int_{(0, \tau)} \chi_D(s) dA(s) = \mathbb{E} \int_{(0, A(\tau-)]} \chi_D(\beta(r)) dr \\ &= \int_{(0, \infty)} \mathbb{P}(r \leq A(\tau), (\beta(r), \omega) \in D) dr. \end{aligned}$$

From this we see that for dr almost all $r \in (0, \infty)$ there is $\Omega(r) \subset \Omega$ with $\mathbb{P}(\Omega(r)) = 1$ such that for any $\omega \in \Omega(r)$ either $r > A(\tau(\omega), \omega)$ or $\beta(r, \omega) < \tau(\omega)$ and for $t = \beta(r)$ and for all $\varphi \in V$

$$(v(t), \varphi) = \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), \varphi \rangle dA(s) + (h(t), \varphi). \quad (4.3)$$

By virtue of Lemma 3.2 there is a nested sequence of decompositions of $[0, 1]$,

$$0 = r_0^n < r_1^n < \dots < r_{N(n)+1}^n = 1, \quad (4.4)$$

such that $\lim_{n \rightarrow \infty} \max_i |r_{j+1}^n - r_j^n| = 0$, and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \int_0^1 \|v(\beta(r)) - v(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} dr &= 0, \\ \lim_{n \rightarrow \infty} \mathbb{E} \int_0^1 \|w_k(\beta(r)) - w_k(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} dr &= 0 \end{aligned} \quad (4.5)$$

for all $i = 1, \dots, m$, all $k \in \mathbb{N}$ and $l = 1, 2$, where $\kappa_n^{(1)}(r) = r_j^n$ if $r \in [r_j^n, r_{j+1}^n)$ and $\kappa_n^{(2)}(r) = r_{j+1}^n$ if $r \in (r_j^n, r_{j+1}^n]$.

Now let $\Omega' := \bigcap_{n \in \mathbb{N}} \left(\Omega(r_0^n) \cap \dots \cap \Omega(r_{N(n)+1}^n) \right)$, $\tau_j^n := \beta(r_j^n)$, and

$$I(\omega) := \{\tau_i^n(\omega) : n \in \mathbb{N}, i = 1, \dots, N(n)\} \cap (0, \infty).$$

Then $\mathbb{P}(\Omega') = 1$ and

$$0 = \tau_0^n < \tau_1^n \leq \tau_2^n \leq \dots \leq \tau_{N(n)+1}^n = \infty, \quad n = 1, 2, \dots,$$

is a nested sequence of random partitions of $(0, 1)$ by stopping times τ_j^n such that statement (1) holds. To prove (2) we notice that, just like in [6], for $r \in (r_j^n, r_{j+1}^n]$

$$v_n^{(2)}(\beta(r)) = \begin{cases} v(\beta(r_{j+1}^n)) = v(\beta(\kappa_n^{(2)}(r))) & \text{if } \beta(r_j^n) < \beta(r) \\ v(\beta(r_j^n)) = v(\beta(\kappa_n^{(1)}(r))) & \text{if } \beta(r_j^n) = \beta(r). \end{cases}$$

Thus with appropriate sets $S_n \in \mathcal{B}(\mathbb{R}) \times \mathcal{F}$

$$v^{(2)}(\beta(r)) = \mathbf{1}_{S_n}(r) v(\beta(\kappa_n^{(2)}(r))) - (1 - \mathbf{1}_{S_n}(r)) v(\beta(\kappa_n^{(1)}(r))).$$

Hence due to (4.5) and Lemma 3.1 we obtain the first equality in (4.2) for $l = 2$, $i = 1, \dots, m$ and for all $k \in \mathbb{N}$. The rest of (4.2) is obtained similarly. \square

Proposition 4.2 *For every $n \in \mathbb{N}$, every $\omega \in \Omega'$ and every $\tau_j^n(\omega) \in I(\omega)$*

$$\begin{aligned} |v(\tau_j^n)|^2 &= |h(0)|^2 + 2 \sum_{i=1}^m \int_{(0, \tau_j^n]} \langle v_i^*(s), v_n^{(2)}(s) \rangle dA(s) \\ &\quad + 2 \int_{(0, \tau_j^n]} \bar{v}_n(s) dh(s) + 2(h(0), h(\tau_1^n) - h(0)) \\ &\quad + \sum_{k=0}^{j-1} |h(\tau_{k+1}^n) - h(\tau_k^n)|^2 - |v(\tau_1^n) - h(\tau_1^n)|^2 \\ &\quad - \sum_{k=1}^{j-1} |v(\tau_{k+1}^n) - v(\tau_k^n) - (h(\tau_{k+1}^n) - h(\tau_k^n))|^2, \end{aligned} \quad (4.6)$$

where $\bar{v}_n(s) = 0$ for $s \in [0, \tau_1^n]$ and $\bar{v}_n(s) = v(\tau_j^n)$ for $s \in (\tau_j^n, \tau_{j+1}^n]$ for $j = 1, \dots, N(n)$. Moreover,

$$\mathbb{E} \sup_{t \in I} |v(t)|^2 < \infty. \quad (4.7)$$

Proof. Let $\omega \in \Omega'$ and $t, t' \in I(\omega)$ and $t' \geq t$. Clearly,

$$|v(t')|^2 - |v(t)|^2 = 2(v(t'), v(t') - v(t)) - |v(t') - v(t)|^2,$$

which by statement (1) of Proposition 4.1 gives

$$\begin{aligned} & |v(t')|^2 - |v(t)|^2 \\ &= 2 \sum_{i=1}^m \int_{(t, t']} \langle v_i^*(s), v(t') \rangle dA(s) + 2(h(t') - h(t), v(t')) - |v(t') - v(t)|^2. \end{aligned}$$

Hence by the identity

$$\begin{aligned} & 2(h(t') - h(t), v(t') - v(t)) \\ &= -|v(t') - v(t) - (h(t') - h(t))|^2 + |v(t') - v(t)|^2 + |h(t') - h(t)|^2, \end{aligned}$$

we have

$$\begin{aligned} |v(t')|^2 - |v(t)|^2 &= 2 \sum_{i=1}^m \int_{(t, t']} \langle v_i^*(s), v(t') \rangle dA(s) + 2(v(t), h(t') - h(t)) \\ &\quad + |h(t') - h(t)|^2 - |v(t') - v(t) - (h(t') - h(t))|^2. \end{aligned} \quad (4.8)$$

By (1) in Proposition 4.1 again

$$2|v(t)|^2 = 2 \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), v(t) \rangle dA(s) + 2(h(t), v(t)),$$

which by the identity $2(h(t), v(t)) = -|v(t) - h(t)|^2 + |v(t)|^2 + |h(t)|^2$ gives

$$|v(t)|^2 = 2 \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), v(t) \rangle dA(s) + |h(t)|^2 - |v(t) - h(t)|^2. \quad (4.9)$$

Summing up for $k = 1, \dots, j-1$ equations (4.8) with $t' = \tau_{k+1}^n$, $t = \tau_k^n$, and adding to it equation (4.9) with $t = \tau_1^n$, we obtain (4.6). From (4.6) we have

$$\begin{aligned} \mathbb{E} \max_{1 \leq j \leq N(n)} |v(\tau_j^n)|^2 &\leq 2\mathbb{E}|h(0)|^2 + 2\mathbb{E} \sum_{i=1}^m \int_{(0, \tau]} |\langle v_i^*(s), v_n^{(2)}(s) \rangle| dA(s) \\ &\quad + 2\mathbb{E} \max_{1 \leq j \leq N(n)} \left| \int_{(0, \tau_j^n]} \bar{v}_n(s) dh(s) \right| \\ &\quad + 2\mathbb{E} \sum_{k=0}^{N(n)} |h(\tau_{k+1}^n) - h(\tau_k^n)|^2. \end{aligned}$$

Clearly

$$2\mathbb{E} \max_{1 \leq j \leq N(n)} \left| \int_{(0, \tau_j^n]} \bar{v}_n(s) dh(s) \right| \leq 16 + \frac{1}{16} \mathbb{E} \sup_{t \geq 0} \left| \int_{(0, t]} \bar{v}_n(s) dh(s) \right|^2,$$

and by Doob's inequality and $\langle h \rangle \leq 1$,

$$\mathbb{E} \sup_{t \geq 0} \left| \int_{(0,t]} \bar{v}_n(s) dh(s) \right|^2 \leq 4\mathbb{E} \int_0^\infty |\bar{v}_n(s)|^2 d\langle h \rangle_s \leq 4\mathbb{E} \max_{1 \leq j \leq N(n)} |v(\tau_j^n)|^2.$$

Since h is a martingale,

$$\mathbb{E} \sum_{k=0}^{N(n)} |h(\tau_{k+1}^n) - h(\tau_k^n)|^2 \leq \mathbb{E} |h(1)|^2 = \mathbb{E} \langle h \rangle(1) \leq 1.$$

By Hölder's inequality and $\sum_i Q_i \leq 1$ we have

$$\begin{aligned} & \sum_{i=1}^m \mathbb{E} \int_{(0,\tau]} |\langle v_i^*(s), v_n^{(2)}(s) \rangle| dA(s) \\ & \leq \sum_i \sup_{n \geq 1} \left(\mathbb{E} \int_{(0,\tau]} \|v_n^{(2)}(s)\|_{V_i}^{p_i} dA(s) \right)^{\frac{1}{p_i}} =: c, \end{aligned}$$

which by virtue of (4.2) is finite. Hence, taking also into account $\mathbb{E} |h(0)|^2 \leq 1$ we have

$$\mathbb{E} \max_{1 \leq j \leq N(n)} |v(\tau_j^n)|^2 \leq 22 + 2c + \frac{1}{4} \mathbb{E} \max_{1 \leq j \leq N(n)} |v(\tau_j^n)|^2,$$

which immediately yields (4.7), provided

$$\mathbb{E} \max_{1 \leq j \leq N(n)} |v(\tau_j^n)|^2 < \infty. \quad (4.10)$$

To show (4.10) note that due to (4.9), for every $n \in \mathbb{N}$ and $j = 1, \dots, N(n)+1$, we get

$$\begin{aligned} \mathbb{E} |v(\tau_j^n)|^2 & \leq \mathbb{E} |h(\tau_j^n)|^2 + 2\mathbb{E} \sum_{i=1}^m \int_{(0,\tau_j^n]} \langle v_i^*(s), v(\tau_j^n) \rangle dA(s) \\ & \leq \mathbb{E} |h(0)|^2 + 2\mathbb{E} \sum_{i=1}^m Q_i(\tau) \left(\int_{(0,\tau]} \|v(\tau_j^n)\|_{V_i}^{p_i} dA(s) \right)^{\frac{1}{p_i}} \\ & \leq 1 + 2 \sum_i \mathbb{E} \|v(\tau_j^n)\|_{V_i}, \end{aligned} \quad (4.11)$$

since $\tau \leq 1$ and $r(t) \leq 1$ for all $t \in [0, \infty)$. For $i = 1, \dots, m$

$$\begin{aligned}
\mathbb{E} \|v(\tau_j^n)\|_{V_i}^{p_i} &\leq \mathbb{E} \sup_{s \in [0, \infty)} \|v_n^{(2)}(s)\|_{V_i}^{p_i} \leq \mathbb{E} \sup_{r \in (0, 1]} \|v_n^{(2)}(\beta(r))\|_{V_i}^{p_i} \\
&\leq 2^{p_i-1} \sum_{l=1}^2 \mathbb{E} \sup_{r \in (0, 1]} \|v(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} \\
&\leq 2^{p_i-1} \sum_{l=1}^2 \mathbb{E} \sum_{k=0}^{N(n)} \frac{1}{r_{k+1}^n - r_k^n} \int_{r_k^n}^{r_{k+1}^n} \|v(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} dr \\
&\leq \frac{2^{p_i-1}}{d_n} \sum_{l=1}^2 \mathbb{E} \int_0^1 \|v(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} dr < 2^{p_i} \frac{c_i}{d_n},
\end{aligned}$$

where r_k^n are given by (4.4), $d_n := \min_{k=1, \dots, N(n)} |r_{k+1}^n - r_k^n| > 0$ and

$$c_i := \max_l \sup_n \sum_{i=1}^m \mathbb{E} \int_0^1 \|v(\beta(\kappa_n^{(l)}(r)))\|_{V_i}^{p_i} dr,$$

which due to (4.5) is finite. Hence by virtue of (4.11) we have (4.10), which completes the proof of (4.7). \square

We see that due to (4.7) there is $\Omega'' \subset \Omega'$ such that $\mathbb{P}(\Omega'') = 1$ and

$$\sup_{t \in I(\omega)} |v(t)|^2 < \infty \quad \text{for all } \omega \in \Omega''.$$

Moreover, since h is cadlag, for all $\omega \in \Omega''$ we have

$$\sup_{t \in I(\omega)} |v(t) - h(t)|^2 < \infty. \quad (4.12)$$

Define

$$z^{(1)}(t) := \int_{(0, t)} \sum_{i=1}^m v_i^*(s) dA(s), \quad z^{(2)}(t) := \int_{(0, t]} \sum_{i=1}^m v_i^*(s) dA(s), \quad (4.13)$$

for $t \geq 0$, where the integrals are defined as weak* integrals. Recall that $v^* = \sum_i v_i^*$ is a V^* -valued such that $\langle v^*(t), \varphi \rangle$ is a progressively measurable process for every $\varphi \in V$, and

$$\begin{aligned}
\int_{(0, t]} |\langle v^*(s), \varphi \rangle| dA(s) &\leq \sum_i \int_{(0, t]} |\langle v_i^*(s), \varphi \rangle| dA(s) \\
&\leq \sum_i |\varphi|_{V_i} \int_{(0, t]} \eta_i(s) dA(s) \leq |\varphi|_V \sum_i \int_{(0, t]} \eta_i(s) dA(s) < \infty.
\end{aligned}$$

Therefore $z^{(1)}$ and $z^{(2)}$ are well-defined V^* -valued processes such that $\langle z^{(1)}, \varphi \rangle$ and $\langle z^{(2)}, \varphi \rangle$ are left-continuous and right-continuous adapted processes, respectively.

In what follows we use the notation $\Delta^w f(t) := f(t) - \text{w-lim}_{s \nearrow t} f(s)$ for H -valued functions f , when the weak limit from the left exists at t .

Proposition 4.3 *Let $z^{(l)}$, $l \in \{1, 2\}$ be given by (4.13).*

1. *If $\omega \in \Omega''$ and $t \in (0, \infty)$ then $z^{(l)}(t) \in H$ for $l \in \{1, 2\}$. Moreover*

$$\sup_{t \in (0, \infty)} |z^{(l)}(t)| < \infty \quad \forall \omega \in \Omega'', \quad l \in \{1, 2\}.$$

2. *Let \tilde{v} be given by*

$$\tilde{v}(t) := \chi_{\Omega''} z^{(2)}(t) + h(t).$$

Then \tilde{v} is a H -valued adapted and weakly cadlag process such that $v(t) = \tilde{v}(t)$ for all $t \in I(\omega)$ and $\omega \in \Omega''$. Moreover

$$\sup_{t \in (0, \infty)} |\tilde{v}(t)| < \infty \quad \forall \omega \in \Omega''.$$

3. *If $\omega \in \Omega''$ then for all $t \in (0, \tau(\omega))$*

$$\Delta^w(\tilde{v} - h)(t) = (\Delta A)(t) \sum_{i=1}^m v_i^*(t). \quad (4.14)$$

Proof. Fix $\omega \in \Omega''$. If $t \in I(\omega)$ then for all $\varphi \in V$

$$(v(t) - h(t), \varphi) = \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), \varphi \rangle dA(s),$$

and hence $z^{(2)}(t) \in H$. Consider now the situation when $t \in (0, \tau(\omega)) \setminus I(\omega)$. Let $\bar{I}^l(\omega)$ denote the left-closure of the set $I(\omega)$. If $t \in \bar{I}^l(\omega) \setminus I(\omega)$ then $\Delta A(t) = 0$ by Proposition 4.1, and there is a sequence $(t_n)_{n \in \mathbb{N}} \subset I(\omega)$ such that $t_n \nearrow t$. Moreover, due to (4.12) there is a subsequence $t_{n'} \nearrow t$ such that $v(t_{n'}) - h(t_{n'})$ converges weakly in H to some $\xi \in H$. Hence for all $\varphi \in V$

$$\begin{aligned} (\xi, \varphi) &= \lim_{n' \rightarrow \infty} (v(t_{n'}) - h(t_{n'}), \varphi) \\ &= \lim_{n' \rightarrow \infty} \sum_{i=1}^m \int_{(0, t_{n'}]} \langle v_i^*(s), \varphi \rangle dA(s) = \sum_{i=1}^m \int_{(0, t)} \langle v_i^*(s), \varphi \rangle dA(s) \\ &= \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), \varphi \rangle dA(s), \end{aligned}$$

which implies $z^{(2)}(t) = \xi \in H$. If $t \in (0, \infty) \setminus \bar{I}^l(\omega)$, then there is $s \in \{0\} \cup \bar{I}^l(\omega)$ such that $s < t$ and $(s, t] \cap I(\omega) = \emptyset$. So $\int_{(s, t]} v_i^*(s) dA(s) = 0$ and $z^{(2)}(t) = z^{(2)}(s) \in H$. Of course if $t = 0$ then $z^{(2)}(t) = 0 \in H$. Finally, due to (4.12),

$$\sup_{t \in (0, \infty)} |z^{(2)}(t)|^2 = \sup_{t \in (0, \infty)} |v(t) - h(t)|^2 < \infty. \quad (4.15)$$

Now we consider $z^{(1)}(t)$ for $t \in (0, \infty)$. Take $(t_n)_{n \in \mathbb{N}}$ such that $t_n < t$ and $t_n \nearrow t$ as $n \rightarrow \infty$. From (4.15) we know that $\sup_{n \in \mathbb{N}} |z^{(2)}(t_n)|^2 < \infty$ and so

there is a subsequence $t_{n'} \nearrow t$ such that $z^{(2)}(t_n)$ converges weakly in H to some $\xi \in H$. Thus for any $\varphi \in V$

$$\begin{aligned} (\xi, \varphi) &= \lim_{n' \rightarrow \infty} (z^{(2)}(t_{n'}), \varphi) \\ &= \lim_{n' \rightarrow \infty} \sum_{i=1}^m \int_{(0, t_{n'}]} \langle v_i^*(s), \varphi \rangle dA(s) = \sum_{i=1}^m \int_{(0, t_{n'}]} \langle v_i^*(s), \varphi \rangle dA(s) = \langle z^{(1)}(t), \varphi \rangle. \end{aligned}$$

Hence $z^{(1)}(t) = \xi \in H$, and due to (4.15)

$$\sup_{t \in (0, \infty)} |z^{(1)}(t)|^2 \leq \sup_{t \in (0, \infty)} |z^{(2)}(t)|^2 < \infty.$$

By construction \tilde{v} is weakly cadlag. Due to (4.15) for $\omega \in \Omega''$

$$\sup_{t \in (0, \infty)} |\tilde{v}(t)|^2 \leq \sup_{t \in (0, \infty)} |z^{(2)}(t)|^2 + \sup_{t \in (0, \infty)} |h(t)|^2 < \infty.$$

We note that for any $\varphi \in V$ the real valued random variable

$$(\tilde{v}(t), \varphi) = \chi_{\Omega''} \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), \varphi \rangle dA(s) + (h(t), \varphi)$$

is \mathcal{F}_t -measurable. Hence, since H is separable, $\tilde{v}(t)$ is \mathcal{F}_t -measurable by the Pettis theorem. Finally notice that

$$\Delta((\tilde{v} - h)(t), \varphi) = \sum_{i=1}^m \langle v_i^*(t), \varphi \rangle (\Delta A)(t)$$

for all $\varphi \in V$ and $\omega \in \Omega''$. Hence on Ω''

$$\Delta^w(\tilde{v} - h)(t) = \sum_{i=1}^m v_i^*(t)(\Delta A)(t).$$

□

Let

$$\tilde{v}_n(t) := \tilde{v}(\tau_j^n) \text{ and } h_n(t) := h(\tau_j^n) \text{ for } t \in (\tau_j^n, \tau_{j+1}^n], \quad j = 0, 1, \dots, N(n).$$

Then from (4.6) it follows that for every $\omega \in \Omega''$ and $t := \tau_j^n(\omega) \in I(\omega)$

$$\begin{aligned} |\tilde{v}(t)|^2 &= |h(0)|^2 + 2 \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), v_n^{(2)}(s) \rangle dA(s) \\ &\quad + 2 \int_{(0, t]} (\tilde{v}_n(s), dh(s)) + \sum_{k=0}^{j-1} |h(\tau_{k+1}^n) - h(\tau_k^n)|^2 - K_n(t), \end{aligned} \tag{4.16}$$

where

$$K_n(t) := \sum_{k: \tau_{k+1}^n \leq t}^{j-1} |\tilde{v}(\tau_{k+1}^n) - \tilde{v}(\tau_k^n) - (h(\tau_{k+1}^n) - h(\tau_k^n))|^2.$$

In order to let $n \rightarrow \infty$ in the above equation we first rewrite it as

$$\begin{aligned} |\tilde{v}(t)|^2 &= 2 \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), v_n^{(2)}(s) \rangle dA(s) \\ &\quad + 2 \int_{(0,t]} (\tilde{v}_n(s) - h_n(s), dh(s)) + |h(t)|^2 - K_n(t) \end{aligned} \quad (4.17)$$

by noticing that

$$2 \int_{(0,\tau_j^n]} (h_n(s), dh(s)) = |h(\tau_j^n)|^2 - |h(0)|^2 - \sum_{k=0}^{j-1} |h(\tau_{k+1}^n) - h(\tau_j^n)|^2.$$

To perform the limit procedure we use the following two propositions.

Proposition 4.4 *There is $\tilde{\Omega} \subset \Omega''$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for a subsequence n' and for every $\omega \in \tilde{\Omega}$*

$$\begin{aligned} \int_{(0,\infty)} \|v(s) - v_{n'}^{(l)}(s)\|_{V_i}^{p_i} dA(s) &\rightarrow 0 \quad (l = 1, 2), \\ \int_{(0,\infty)} \|w_k(s) - w_{kn'}^{(l)}(s)\|_{V_i}^{p_i} dA(s) &\rightarrow 0 \quad (l = 1, 2; k \in \mathbb{N}), \\ \sup_{t \in (0,\infty)} \left| \int_{(0,t]} (\tilde{v}_{n'}(s) - h_{n'}(s), dh(s)) - \int_{(0,t]} (\tilde{v}(s-) - h(s-), dh(s)) \right| &\rightarrow 0 \end{aligned}$$

as $n' \rightarrow \infty$. Moreover,

$$K_{n'}(t) \rightarrow \int_{(0,t]} |v^*(s)|^2 \Delta A(s) dA(s) \quad \text{for } t \in I(\omega) \text{ and } \omega \in \tilde{\Omega}.$$

Proof. Set $\xi(t) := \tilde{v}(t-) - h(t-)$ and $\xi_n(t) := \tilde{v}_n(t) - h_n(t)$. By Lemma 3.3, taking into account that by Proposition 4.3 on Ω''

$$\sup_n \sup_{t \in (0,\infty)} |\xi(t) - \xi_n(t)| \leq \sup_{t \in (0,\infty)} |z^{(1)}(t)| < \infty,$$

and that V is dense in H , we have

$$\sup_{t \geq 0} \left| \int_{(0,t]} (\xi(s) - \xi_n(s), dh(s)) \right| \rightarrow 0 \quad \text{in probability as } n \rightarrow \infty,$$

if we show that almost surely

$$\lim_{n \rightarrow \infty} (\xi(t) - \xi_n(t), \varphi) = 0 \quad \text{for all } t > 0 \text{ and } \varphi \in V.$$

To this end set

$$v_i^* := \int_{(\tau_j^n, t)} v_i^*(s) dA(s) \in V_i^*.$$

Then for all $\omega \in \Omega''$, $t > 0$ and $\varphi \in V$

$$\begin{aligned}
(\xi(t) - \xi_n(t), \varphi) &= \langle \xi(t) - \xi_n(t), \varphi \rangle = \left\langle \sum_{i=1}^m v_i^*, \varphi \right\rangle = \sum_{i=1}^m \langle v_i^*, \varphi \rangle \\
&= \sum_{i=1}^m \int_{(\tau_j^n, t)} \langle v_i^*(s), \varphi \rangle dA(s) \leq \sum_{i=1}^m \|\varphi\|_{V_i} \int_{(\tau_j^n, t)} \|v_i^*(s)\|_{V_i^*} dA(s) \\
&\leq \max_{j=1, \dots, N(n)} \sum_{i=1}^m \|\varphi\|_{V_i} (A(\tau_{j+1}^n) - A(\tau_j^n))^{\frac{1}{p_i}} Q_i(\tau_{j+1}^n) \\
&\leq \max_{j=1, \dots, N(n)} \sum_{i=1}^m \|\varphi\|_{V_i} |r_{j+1}^n - r_j^n|^{\frac{1}{p_i}} \rightarrow 0 \text{ as } n \rightarrow \infty,
\end{aligned}$$

with r_j^n given by (4.4). Consequently, taking also into account (4.2) of Proposition 4.1 we have $\Omega''' \subset \Omega''$ and a subsequence n' such that the first three limits are zero for $\omega \in \Omega'''$. Taking the limit along the subsequence n' in (4.17) we see that $K_{n'}(t)$ converges for $\omega \in \Omega'''$ and $t \in I(\omega)$ to some $K(t)$, and

$$\begin{aligned}
|\tilde{v}(t)|^2 &= 2 \sum_{i=1}^m \int_{(0, t]} \langle v_i^*(s), v(s) \rangle dA(s) \\
&\quad + 2 \int_{(0, t]} (\tilde{v}(s) - h(s), dh(s)) + |h(t)|^2 - K(t).
\end{aligned}$$

From this point onwards we will always consider only the subsequence n' but we will keep writing n to ease notation. Our task is now to identify $K(t)$. We note that, using Parseval's identity,

$$\begin{aligned}
K_n(t) &= \sum_{0 \leq \tau_{j+1}^n \leq t} \left| \sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s) \right|^2 \\
&= \sum_{0 \leq \tau_{j+1}^n \leq t} \sum_{k \in \mathbb{N}} \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s), e_k \right)^2 \\
&= \sum_{0 \leq \tau_{j+1}^n \leq t} \sum_{k \in \mathbb{N}} \left\langle \sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s), e_k \right\rangle^2 \\
&= \sum_{0 \leq \tau_{j+1}^n \leq t} \sum_{k \in \mathbb{N}} \left| \int_{(\tau_j^n, \tau_{j+1}^n]} \sum_i \langle v_i^*(s), e_k \rangle dA(s) \right|^2.
\end{aligned}$$

Hence, using Lemma 3.1, Parseval's identity and (4.14), we get

$$\begin{aligned}
K(t) &= \lim_{n \rightarrow \infty} K_n(t) \geq \sum_{k \in \mathbb{N}} \lim_{n \rightarrow \infty} \sum_{0 \leq \tau_{j+1}^n \leq t} \left| \int_{(\tau_j^n, \tau_{j+1}^n]} \sum_i \langle v_i^*(s), e_k \rangle dA(s) \right|^2 \\
&= \sum_{k \in \mathbb{N}} \sum_{s \leq t} \left| \sum_i \langle v_i^*(s), e_k \rangle \Delta A(s) \right|^2 = \sum_{k \in \mathbb{N}} \sum_{s \leq t} |(\Delta^w(\tilde{v} - h)(s), e_k)|^2 \\
&= \sum_{s \leq t} \left| \sum_i v_i^*(s) \right|^2 |\Delta A(s)|^2. \tag{4.18}
\end{aligned}$$

To obtain an upper bound we use first the identity

$$|x + y|^2 = y^2 + 2x(y + x) - x^2$$

together with the definition of g to get

$$\begin{aligned}
K_n(t) &= \sum_{0 \leq \tau_{j+1}^n \leq t} \left| \sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s) + \sum_i v_i^*(\tau_{j+1}^n) \Delta A(\tau_{j+1}^n) \right|^2 \\
&= \sum_{0 \leq \tau_{j+1}^n \leq t} (J_j^{(1)} + J_j^{(2)} - J_j^{(3)}) \tag{4.19}
\end{aligned}$$

with

$$\begin{aligned}
J_j^{(1)} &:= \left| \sum_i v_i^*(\tau_{j+1}^n) \right|^2 |\Delta A(\tau_{j+1}^n)|^2, \\
J_j^{(2)} &:= 2 \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s), \tilde{v}(\tau_{j+1}^n) - \tilde{v}(\tau_j^n) - (h(\tau_{j+1}^n) - h(\tau_j^n)) \right), \\
J_j^{(3)} &:= \left| \sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s) \right|^2
\end{aligned}$$

For $j \neq 0$ we split $J_j^{(2)} = J_j^{(21)} - J_j^{(22)}$ with

$$\begin{aligned}
J_j^{(21)} &:= 2 \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s), \tilde{v}(\tau_{j+1}^n) - \tilde{v}(\tau_j^n) \right), \\
J_j^{(22)} &:= 2 \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} v_i^*(s) dA(s), h(\tau_{j+1}^n) - h(\tau_j^n) \right),
\end{aligned}$$

and notice that

$$J_j^{(21)} = 2 \sum_i \int_{(\tau_j^n, \tau_{j+1}^n]} \langle v_i^*(s), v(\tau_{j+1}^n) - v(\tau_j^n) \rangle dA(s)$$

$$= 2 \sum_i \int_{[\tau_j^n, \tau_{j+1}^n)} \langle v_i^*(s), v_n^{(2)}(s) - v_n^{(1)}(s) \rangle dA(s).$$

Using Π_k , the orthogonal projection of H onto the space spanned by $(e_j)_{j=1}^k \subset V$, we have

$$J_j^{(22)} = J_{jk}^{(22)} + \bar{J}_{jk}^{(22)}$$

with

$$J_{jk}^{(22)} := 2 \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n)} v_i^*(s) dA(s), \Pi_k(h(\tau_{j+1}^n) - h(\tau_j^n)) \right),$$

$$\bar{J}_{jk}^{(22)} := 2 \left(\sum_i \int_{(\tau_j^n, \tau_{j+1}^n)} v_i^*(s) dA(s), (I - \Pi_k)(h(\tau_{j+1}^n) - h(\tau_j^n)) \right).$$

Notice that

$$J_{jk}^{(22)} = 2 \sum_i \int_{(\tau_j^n, \tau_{j+1}^n)} \langle v_i^*(s), \Pi_k(h(\tau_{j+1}^n) - h(\tau_j^n)) \rangle dA(s),$$

$$= 2 \sum_i \int_{[\tau_j^n, \tau_{j+1}^n)} \langle v_i^*(s), w_{kn}^{(2)}(s) - w_{kn}^{(1)}(s) \rangle dA(s),$$

and

$$\bar{J}_{jk}^{(22)} \leq J_j^{(3)} + |(I - \Pi_k)(h(\tau_{j+1}^n) - h(\tau_j^n))|^2.$$

Similarly, taking into account $\tilde{v}(0) = h(0)$, for $J_0^{(2)}$ we have

$$J_0^{(2)} = 2 \left(\sum_i \int_{(0, \tau_1^n)} v_i^*(s) dA(s), \tilde{v}(\tau_1^n) - h(\tau_1^n) \right) = J_0^{(21)} - J_0^{(22)},$$

where

$$J_0^{(21)} := 2 \sum_i \int_{(0, \tau_1^n)} \langle v_i^*(s), \tilde{v}(\tau_1^n) \rangle dA(s)$$

$$= 2 \sum_i \int_{(0, \tau_1^n)} \langle v_i^*(s), v^{(2)}(s) - v^{(1)}(s) \rangle dA(s),$$

and

$$J_0^{(22)} := 2 \left(\sum_i \int_{(0, \tau_1^n)} v_i^*(s) dA(s), h(\tau_1^n) \right) = J_{0k}^{(22)} + \bar{J}_{0k}^{(22)}$$

with

$$J_{0k}^{(22)} := \sum_i \int_{(0, \tau_1^n)} \langle v_i^*(s), w^{(2)}(s) - w^{(1)}(s) \rangle dA(s),$$

$$\bar{J}_{0k}^{(22)} := 2 \left(\sum_i \int_{(0, \tau_1^n)} v_i^*(s) dA(s), (I - \Pi_k)h(\tau_1^n) \right)$$

$$\leq J_0^{(3)} + |(I - \Pi_k)h(\tau_1^n)|^2.$$

Thus from (4.19) we get

$$\begin{aligned} K_n(t) &\leq \sum_{0 \leq \tau_{j+1}^n \leq t} \left| \sum_i v_i^*(\tau_{j+1}^n) \right|^2 |\Delta A(\tau_{j+1}^n)|^2 \\ &\quad + 2 \sum_i \int_{(0,t)} \langle v_i^*(s), v_n^{(2)}(s) - v_n^{(1)}(s) \rangle dA(s) \\ &\quad - 2 \sum_i \int_{(0,t)} \langle v_i^*(s), w_{nk}^{(2)}(s) - w_{nk}^{(1)}(s) \rangle dA(s) + \xi_{nk}(t) \end{aligned}$$

with

$$\xi_{nk}(t) := \sum_{j=1}^{N(n)} \left| (I - \Pi_k)(h(\tau_{j+1}^n \wedge t) - h(\tau_j^n \wedge t)) \right|^2 + |(I - \Pi_k)h(\tau_1^n \wedge t)|^2$$

for every $n, k \in \mathbb{N}$. As $n \rightarrow \infty$ we see that

$$\sum_{0 \leq \tau_{j+1}^n \leq t} \left| \sum_i v_i^*(\tau_{j+1}^n) \right|^2 |\Delta A(\tau_{j+1}^n)|^2 \rightarrow \sum_{0 < s \leq t} |v^*(s)|^2 |\Delta A(s)|^2,$$

where we use the notation $v^*(s) = \sum_i v_i^*(s)$. By Hölder's inequality, taking into account $r(t) \leq 1$, we have

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \int_{(0,t)} |\langle v_i^*(s), v_n^{(2)}(s) - v_n^{(1)}(s) \rangle| dA(s) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left(\int_{(0,t)} \|v_n^{(2)}(s) - v_n^{(1)}(s)\|_{V_i}^{p_i} dA(s) \right)^{\frac{1}{p_i}} = 0 \end{aligned}$$

and similarly,

$$\overline{\lim}_{n \rightarrow \infty} \int_{(0,t)} |\langle v_i^*(s), w_{nk}^{(2)}(s) - w_{nk}^{(1)}(s) \rangle| dA(s) = 0$$

for all integers $k \geq 1$ and $i = 1, 2, \dots, m$. Thus

$$K(t) = \varliminf_{n \rightarrow \infty} K_n(t) \leq \sum_{s \leq t} |v^*(s)|^2 |\Delta A(s)|^2 + \xi_k(t) \quad (4.20)$$

for every $k \in \mathbb{N}$, where

$$\xi_k(t) := \varliminf_{n \rightarrow \infty} \left(\sum_{j=1}^{N(n)} \left| (I - \Pi_k)(h(\tau_{j+1}^n \wedge t) - h(\tau_j^n \wedge t)) \right|^2 + |(I - \Pi_k)h(\tau_1^n \wedge t)|^2 \right).$$

Note that by Fatou's lemma and the martingale property of h

$$\mathbb{E}\xi_k(t) \leq \mathbb{E}|(I - \Pi_k)h(1)|^2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Note also that for each $\omega \in \Omega$ and $t \in (0, \infty)$ we have $\xi_k \geq \xi_{k+1}$ and $\xi_k \geq 0$. Thus there exists a set $\Omega'''' \subset \Omega$ with $P(\Omega''') = 1$ such that for every $t \in [0, \infty)$ and $\omega \in \Omega''''$ we have $\xi_k(t) \rightarrow 0$. Letting here $k \rightarrow \infty$ in (4.20) we obtain

$$K(t) \leq \int_{(0,t]} |v^*(s)|^2 \Delta A(s) dA(s),$$

which together with (4.18) gives

$$K(t) = \int_{(0,t]} |v^*(s)|^2 \Delta A(s) dA(s)$$

for $\omega \in \tilde{\Omega} := \Omega''' \cap \Omega''''$ and $t \in I(\omega)$. □

Proposition 4.5 For $\omega \in \tilde{\Omega}$

$$\begin{aligned} |\tilde{v}(t)|^2 &= |h(0)|^2 + 2 \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), v(s) \rangle dA(s) \\ &+ 2 \int_{(0,t]} (\tilde{v}(s-), dh(s)) - \int_{(0,t]} \left| \sum_{i=1}^m v_i^*(s) \right|^2 \Delta A(s) dA(s) + [h]_t \end{aligned} \quad (4.21)$$

for $t \in [0, \tau(\omega))$.

Proof. Let $\omega \in \tilde{\Omega}$ be fixed and let $t \in [0, \tau(\omega))$. To ease notation we use $n \rightarrow \infty$ in place of the subsequence $n' \rightarrow \infty$ defined in the previous proposition. If $t \in I(\omega)$, then by virtue of the previous proposition taking $n \rightarrow \infty$ in (4.17) we obtain

$$\begin{aligned} |\tilde{v}(t)|^2 &= |h(t)|^2 + 2 \sum_{i=1}^m \int_{(0,t]} \langle v_i^*(s), v(s) \rangle dA(s) + 2 \int_{(0,t]} (\tilde{v}(s-) - h(-), dh(s)) \\ &- \int_{(0,t]} \left| \sum_{i=1}^m v_i^*(s) \right|^2 \Delta A(s) dA(s). \end{aligned}$$

Hence using the Itô formula for Hilbert space valued processes

$$|h(t)|^2 = |h(0)|^2 + 2 \int_{(0,t]} (h(s-), dh(s)) + [h]_t, \quad (4.22)$$

we get (4.21) for $t \in I(\omega)$. If $t \in \bar{I}^l(\omega) \setminus I(\omega)$, then for sufficiently large n there is $j = j(n)$ such that $t_n := \tau_j^n(\omega) \in I(\omega)$ and $t_n \nearrow t$ for $n \rightarrow \infty$. Using the algebraic relationship

$$|\tilde{v}(s) - \tilde{v}(r)|^2 = |\tilde{v}(s)|^2 - |\tilde{v}(r)|^2 - 2(\tilde{v}(r), \tilde{v}(s) - \tilde{v}(r)), \quad (4.23)$$

with $s := t_n$, $r := t_l$, and since (4.21) holds for every $t \in I(\omega)$, we get

$$\begin{aligned} |\tilde{v}(t_n) - \tilde{v}(t_l)|^2 &= 2 \sum_{i=1}^m \int_{(t_l, t_n]} \langle v_i^*(s), v(s) \rangle dA(s) + 2 \int_{(t_l, t_n]} (\tilde{v}(s-) , dh(s)) \\ &\quad - \int_{(t_l, t_n]} \left| \sum_{i=1}^m v_i^*(s) \right|^2 \Delta A(s) dA(s) + [h]_{t_n} - [h]_{t_l} - 2(\tilde{v}(t_l), \tilde{v}(t_n) - \tilde{v}(t_l)) \end{aligned}$$

for $n > l$. Moreover

$$\begin{aligned} &2(\tilde{v}(t_l), \tilde{v}(t_n) - \tilde{v}(t_l)) \\ &= 2 \sum_{i=1}^m \int_{(t_l, t_n]} \langle v_i^*(s), v(t_l) \rangle dA(s) + 2(\tilde{v}(t_l), h(t_n) - h(t_l)). \end{aligned}$$

Hence by (4.22)

$$\begin{aligned} |\tilde{v}(t_n) - \tilde{v}(t_l)|^2 &= 2 \sum_{i=1}^m \int_{(t_l, t_n]} \langle v_i^*(s), v(s) - v(t_l) \rangle dA(s) \\ &\quad + 2 \int_{(t_l, t_n]} (\tilde{v}(s-) - h(s-) - (\tilde{v}(t_l) - h(t_l)), dh(s)) \\ &\quad - \int_{(t_l, t_n]} \left| \sum_{i=1}^m v_i^*(s) \right|^2 \Delta A(s) dA(s) + |h(t_n) - h(t_l)|^2 \\ &=: 2I_{ln}^1 + 2I_{ln}^2 - I_{ln}^3 + I_{ln}^4. \end{aligned}$$

Since h is cadlag we have

$$\lim_{l \rightarrow \infty} \sup_{n > l} I_{ln}^4 = \lim_{l \rightarrow \infty} \sup_{n > l} |h(t_n) - h(t_l)|^2 = 0.$$

By the previous proposition we get

$$\begin{aligned} \lim_{l \rightarrow \infty} \sup_{n > l} |I_{ln}^2| &= \lim_{l \rightarrow \infty} \sup_{n > l} \left| \int_{(t_l, t_n]} (\tilde{v}(s-) - h(s-) - (\tilde{v}_l(s) - h_l(s)), dh(s)) \right| \\ &\leq 2 \lim_{l \rightarrow \infty} \sup_{t \in (0, \infty)} \left| \int_{(0, t]} (\tilde{v}(s-) - h(s-) - (\tilde{v}_l(s) - h_l(s)), dh(s)) \right| = 0, \end{aligned}$$

and

$$\lim_{l \rightarrow \infty} \sup_{n > l} |I_{ln}^1| \leq \lim_{l \rightarrow \infty} \sum_{i=1}^m \int_{(0, \infty)} \|v_i^*(s)\|_{V_i^*} \|v(s) - v_l^{(1)}(s)\|_{V_i} dA(s) = 0,$$

via $r(t) \leq 1$ and Hölder's inequality. Thus

$$\lim_{l \rightarrow \infty} \sup_{n > l} |\tilde{v}(t_n) - \tilde{v}(t_l)|^2 = 0,$$

and so the sequence $(\tilde{v}(t_n))_{n \in \mathbb{N}}$ converges strongly to some ξ in H . Moreover since \tilde{v} is weakly cadlag and $t_n \nearrow t$, we conclude that $\xi = \tilde{v}(t-)$. Hence using (4.21) with t_n in place of t , and letting $n \rightarrow \infty$ we obtain

$$\begin{aligned} |\tilde{v}(t-)|^2 &= |h(0)|^2 + 2 \sum_{i=1}^m \int_{(0,t)} \langle v_i^*(s), v(s) \rangle dA(s) \\ &\quad + 2 \int_{(0,t)} (\tilde{v}(s-), dh(s)) - \int_{(0,t)} \left| \sum_{i=1}^m v_i^*(s) \right|^2 \Delta A(s) dA(s) + [h]_{t-} \end{aligned}$$

for $t \in I(\omega) \setminus \bar{I}^l(\omega)$, and so for this t we get also (4.21) by taking into account that $\Delta A(t) = 0$. If $t \in (0, \tau(\omega)) \setminus \bar{I}^l(\omega)$, then there is $t' \in \{0\} \cup \bar{I}^l(\omega)$ such that $t' < t$ and $(t', t] \cap I(\omega) = \emptyset$. Thus $dA(s) = 0$ for $s \in (t', t]$, and so $\tilde{v}(s) - \tilde{v}(t') = h(s) - h(t')$. Hence applying (4.21) with $t := t'$, and the formula

$$|\tilde{v}(t)|^2 - |\tilde{v}(t')|^2 = 2(\tilde{v}(t'), \tilde{v}(t) - \tilde{v}(t')) + |\tilde{v}(t) - \tilde{v}(t')|^2$$

together with the Itô formula for Hilbert space valued martingales,

$$|h(t) - h(t')|^2 = 2 \int_{(t', t]} (h(s-) - h(t'), dh(s)) + [h]_t - [h]_{t'},$$

we obtain (4.21) for the t under consideration. \square

Now we can finish the proof of Theorem 2.1 by noting that by the above proposition $|\tilde{v}(t)|^2$ is a cadlag process, and since by Proposition 4.3 the process \tilde{v} is H -valued and weakly cadlag, it follows by identity (4.23) that \tilde{v} is an H -valued cadlag process.

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